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Market Risk Modelling in the Solvency II Regime and Hedging Options Without Using the Underlying

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Abstract: In the paper mathematical tools for quantile hedging in an incomplete market are developed. Those could be used for two significant applications. The first one is calculating the optimal capital requirement imposed by Solvency II when the market and non-market risks are present in an insurance company. We show how to find the minimal capital V_0 to provide the one-year hedging strategy for an insurance company satisfying $\mathbb{E}[\mathbf{1}_{\{V_1 \geq D\}}] = \mathbf{0.995}$, where V_1 denotes the value of the company's assets after one year, and D is the payoff of the contract. The second application is to find a hedging strategy for a derivative that does not use the underlying asset itself, but another asset whose dynamics are correlated with – or otherwise stochastically dependent on – those of the underlying. The paper generalises the results of obtained by Klusik and Palmowski in 2011.

Keywords: quantile hedging, Solvency II, capital modelling, hedging options on a non-tradable asset

1. Introduction

Directive of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II) introduces new capital regimes on insurance companies (Directive 2009/138/EC…). According to Chapter 6, Section 4, Art. 101, p. 3: "The Solvency Capital Requirement shall be calibrated so as to ensure that all quantifiable risks to which an insurance or reinsurance undertaking is exposed are taken into account. It shall cover existing business, as well as the new business expected to be written over the following 12 months. With respect to existing business, it shall cover only unexpected losses.

It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5 % over a one-year period."

Further according to p. 4: "The Solvency Capital Requirement shall cover at least the following risks:

- non-life underwriting risk,
- life underwriting risk,
- health underwriting risk,
- market risk,
- credit risk,
- operational risk."

The key question raised by this regulation is: how much capital is sufficient to hedge against risks with a 99.5% probability? What is important here from the mathematical point of view, is that the risk involves market and nonmarket factors, which means that it cannot be dealt using just the real expectations probability measure. Insurance companies usually neglect this, although this neglect opposes the widely accepted Black-Scholes approach.

Mathematically speaking, we ask for minimal V_0 ensuring the probability of satisfying all the claims $\mathbb{E}\left[\mathbf{1}_{\{V_1 \geq D\}}\right] \geq 0.995$, where D denotes the contingent claim and V_t denotes the value of the hedging portfolio at time t. Equivalently, we can fix the capital and look for a strategy with the maximal probability of a successful hedging.

This problem was solved in literature only for complete markets (besides Sekine (2000) and Klusik & Palmowski (2011)), i.e. for financial positions which do not allow for typical insurance risk.

Föllmer & Leukert (1999) investigated the general semimartingale setting. They pointed out the optimal strategy for a complete market with maximal $E[\mathbf{1}_{\{V_1 \geq D\}}]$. The proofs are based on various versions of the Neyman-Pearson lemma. Spivak & Cvitanić (1999) studied a complete market framework of assets modelled with Ito processes. They also constructed a strategy with maximal $E[\mathbf{1}_{\{V_1 \geq D\}}]$ but using different proof methods. They also implemented this technique for a market with partial observations. Finally they considered the case where the drift of the wealth process was a nonlinear (concave) function of the investment strategy of the agent.

Klusik, Palmowski, and Zwierz (2010) solved the problem of the quantile hedging from the point of view of a better informed agent acting on the market. The additional knowledge of the agent was modelled by a filtration initially enlarged by some random variables.

Sekine (2000) considered defaultable securities in a very simple incomplete market, where a security-holder could default at some random times and received a payoff modelled by a martingale process. The author showed a strategy maximising the probability of a successful hedge.

A more complex incomplete market was studied by Klusik and Palmowski (2011). They considered the equity-linked product where the insurance event could take a finite number of states and was independent of a financial asset modelled by a geometric Brownian motion. They constructed the optimal strategy for both: the maximal probability and the maximal expected success ratio. In their framework the knowledge about the insurance event was not revealed before the maturity.

In this paper we state a general problem of optimising probability of non-insolvency $\mathbb{E}[\mathbf{1}_{\{V_1 \geq D\}}]$ in an incomplete market, as in (Klusik & Palmowski, 2011), but we allow a very general flow of information outside the market and a very general space of possible non-market events. As it was said at the beginning the solution of this problem gives a solution to Solvency II problem.

In fact, the solution extends beyond Solvency II and applies to pricing instruments in incomplete markets. These include equity-linked instruments or options on illiquid assets traded exclusively

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over-the-counter. In such scenarios, constructing a replicating strategy is often infeasible. For comparison, in complete markets, the pricing process is significantly simpler, as the instrument's price equals the value of a replicating portfolio.

Conversely, for non-replicable instruments found in incomplete markets (e.g., equity-linked products), such portfolios do not exist. To achieve full protection, one must construct a portfolio capable of hedging against highly expensive and practically improbable worst-case scenarios. This is known as superhedging – a strategy designed to handle worst-case outcomes, albeit with prohibitive costs in practice. Alternatively, the option price is very often calculated as an expectation under a subjective probability measure, which, while typically appropriate in insurance contexts, is often flawed in financial theory when market risks tied to stock exchanges are present.

From a practical perspective, many institutions rely on a straightforward approach to hedge financial instruments, employing a correlated liquid asset in place of the underlying asset, based on the intuition that this method is sufficiently effective. The main advantage of this approach is its simplicity, avoiding the need for complex quantitative analysis while maintaining the belief that it adequately mitigates risk in most practical scenarios. However, this approach has significant drawbacks, including a lack of rigorous quantitative control, a reliance on potentially flawed intuition, and the susceptibility of various risk indicators to human error.

This paper is organized as follows. Section 2 introduces the financial market model and the associated optimization problem. We also define and provide the price of the hedging strategy.

In section 3, we present the application of our result for hedging the European put option on a non-tradable asset. We calculate the cost of a hedging strategy using another asset whose price process is partially dependent on the underlying. In the numerical calculation, we assume that both price processes are driven by correlated geometric Brownian motions.

2. Mathematical Model

Consider a discounted price process $\mathbb{X} = (X_t)_{t\in[0,T]}$ which is a semimartingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}_\mathrm{T} = \mathcal{F}$. Note that $\mathbb{F}$ may be substantially greater than filtration generated by $\mathbb{X}$. The interpretation is as follows: the knowledge modelled by $\mathbb{F}$ could be augmented by information from outside the market. The augmentation of filtration could be interpreted as the information signal about non-market variables affecting the value of a contract. An example: the 'life' part of information about the equity-linked contract. We will assume that $\mathcal{F} = \mathcal{F}_\mathrm{T}$.

Denote the set of all equivalent martingale measures by $\mathcal{P}$ and assume that the market does not allow for arbitrage, i.e. $\mathcal{P} \neq \emptyset$.

A self-financing admissible trading strategy is a pair (V_0, ξ), where V_0 is a constant and ξ is an $\mathbb{F}$-predictable process on $[0, T]$ for which the value process $V_t \coloneqq V_0 + \int_0^t \xi_u dX_u$, $t \in [0, T]$, is well defined and $V_t \geq 0$ $\mathbb{P}$-almost surely for all $t \in [0, T]$.

Fix an initial capital $\tilde{V}_0$ and denote by $\mathcal{A}$ the set of all admissible strategies (V_0, ξ) such that $V_0 \leq \tilde{V}_0$.

For nonnegative real v, d define a *success factor* ϕ_d^v assuming values in [0,1] such that ϕ_d^v is a nondecreasing function of v for every (or each) d. The following functions can serve as examples of the success factor: $\phi_d^v := 1_{\{v\geq d\}}$ and $\phi_d^v := 1_{\{v\geq d\}} + 1_{\{v<d\}}\frac{v}{d}$.

For a contingent claim D being an $\mathcal{F}_T$-measurable nonnegative random variable we formulate the following problem:

Problem 2.1. Find $(V_0, \xi) \in \mathcal{A}$ maximising the expected *success factor* $\mathbb{E}^{\mathbb{P}}[\phi_D^{V_T}]$.

Before solving this problem, we introduce the following notation: For any increasing function $g: [0, \infty) \to \mathbb{R}$ and positive constant m:

$$\pi_m^g := \min\{\arg\min_{x \geq 0}(mx - g(x))\}\,. \tag{2.1}$$

In the event that $mx - g(x)$ attains its minimum at multiple points, we define π_m^g as the smallest (or first) such point, in order to avoid ambiguity.

From this definition we see that for $x \geq 0$:

$$mx - g(x) \geq m\pi_m^g - g(\pi_m^g), \text{ i.e.}$$

$$\hat{g}_m(x) := m\left(x - \pi_m^g\right) + g(\pi_m^g) \geq g(x)$$

and equality holds at $x = \pi_m^g$. Consequently, $\hat{g}_m$ can be viewed as the line with slope m that 'touches' g from above at $x = \pi_m^g$ (see Fig. 1).

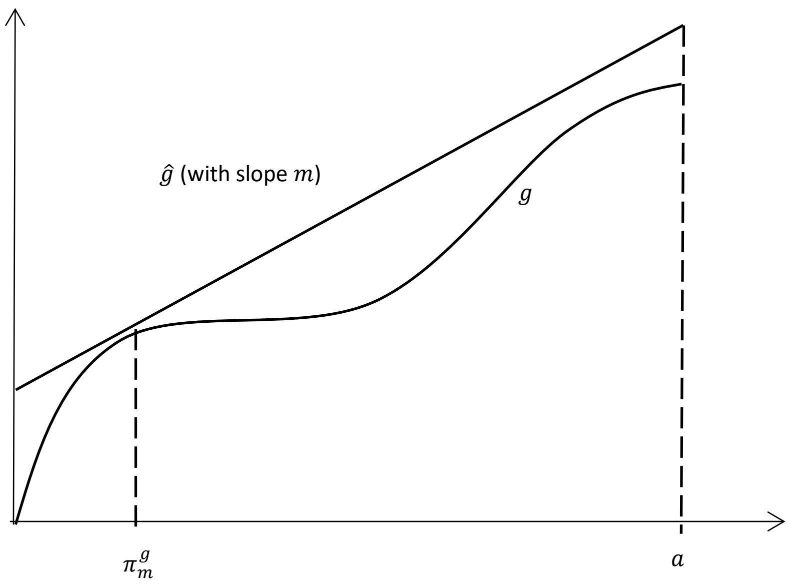

Figure 1. The picture shows the relation between g, $\hat{g}_m$, m and π_m^g

Source: own elaboration.

Fix a measure $\mathbb{Q} \in \mathcal{P}$ and define

$$G_{\mathbb{Q}}(x) := \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\phi_D^x | \mathbb{X}\right]. \tag{2.2}$$

Note that, for almost every ω, the function $\boldsymbol{x} \mapsto G_{\mathbb{Q}}(x)(\omega)$ is increasing in x.

Assume that there is a positive constant m that $\pi_m^{G_{\mathbb{Q}}}$ exists for almost every ω, and that the random variable $\pi_m^{G_{\mathbb{Q}}}$ is replicable with a strategy (V_0^*, ξ^*) where $V_0^* = \tilde{V}_0$. That is,

$$\pi_m^{G_{\mathbb{Q}}} = V_0^* + \int_0^T \xi_u^* dX_u. \tag{2.3}$$

Remark 2.1. Note that the above assumption is always satisfied if all measures in $\mathcal{P}$ coincide on $\sigma(\mathbb{X})$. This holds because $\pi_m^{G_{\mathbb{Q}}}$ is $\sigma(\mathbb{X})$-measurable. In particular, this applies to the complete-market case extended with contingent claims dependent on a randomness from outside the market (also in a situation when the information outside the market is revealed continuously till the moment of maturity). In the next section we give a numerical procedure to estimate m by the Monte Carlo simulation.

From now on, we write π as a shortcut to $\pi_m^{G_{\mathbb{Q}}}$.

Theorem 2.1. (V_0^*, ξ^*) is a solution to Problem 2.1 with the expected success factor equal $\mathbb{E}^{\mathbb{P}}[\phi_D^\pi]$.

Proof. For any $(V_0, \xi) \in \mathcal{A}$ holds $\mathrm{E}^{\mathbb{Q}}[V_T - \pi] \leq V_0 - \tilde{V}_0 \leq 0$.

For every x the inequality $\hat{G}_{\mathbb{Q}}^m(x) \geq G_{\mathbb{Q}}(x)$ holds a.s., where $\hat{G}_{\mathbb{Q}}^m(x) = G_{\mathbb{Q}}(\pi) + m(x - \pi)$. Thus $\mathbb{E}^{\mathbb{Q}}[\hat{G}_{\mathbb{Q}}^m(x)] \geq \mathbb{E}^{\mathbb{Q}}[G_{\mathbb{Q}}(x)]$, i.e.

$$\mathbb{E}^{\mathbb{P}}\left[\phi_D^{V_T^*}\right] = \mathbb{E}^{\mathbb{P}}\left[\phi_D^{\pi}\right] \geq \mathbb{E}^{\mathbb{P}}\left[\phi_D^{\pi}\right] + m\mathbb{E}^{\mathbb{Q}}\left[V_T - \pi\right] \geq \mathbb{E}^{\mathbb{P}}\left[\phi_D^{V_T}\right]. \tag{2.4}$$

Note that $(V_0^*, \xi^*) \in \mathcal{A}$ because $V_0^* = \tilde{V}_0$, so the left side of the inequality is attainable.

3. Applications

3.1. Hedging Contingent Without Underlying

We consider a situation where we sell a European put option on non-tradable asset Y with the payoff $D = (K - Y_T)^+$. We are going to hedge it using tradable asset X with the strategy maximising $\mathbb{P}(V_T \geq D)$. Assume that the dynamics of two price processes are given by the following equations

$$dX_t = \mu_X X_t dt + \sigma_X X_t dW_t^X, X_0 = x_0 > 0,$$

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y, Y_0 = y_0 > 0, \tag{3.1}$$

where we assume a correlation ρ between two Brownian motions W^Y and W^X, i.e. $W^Y = \rho W^X + \sqrt{1-\rho^2}W$ where W is a Brownian motion independent of W^X. We assume that the interest rate is equal to zero. For 0 < *x* < *D* we have

$$\begin{aligned}
& G_{\mathbb{Q}}(x) \\
= \; & \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}} 1_{\{D \leq x\}} \middle| \mathbb{X}\right] = \frac{d\mathbb{P}}{d\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[1_{\{(K-Y_T)^+ \leq x\}} \middle| \mathbb{X}\right] \\
= \; & \frac{d\mathbb{P}}{d\mathbb{Q}} \mathbb{Q}\left[K - x \leq y_0 \exp\left\{\mu_Y T + \sigma_Y(\rho W_T^X + \sqrt{1-\rho^2} W_T) - \frac{1}{2}\sigma_Y^2 T\right\} \middle| \mathbb{X}\right] \\
= \; & \frac{d\mathbb{P}}{d\mathbb{Q}} \mathbb{Q}\left[\ln\left(\frac{K-x}{y_o}\right) \leq \mu_Y T + \sigma_Y(\rho W_T^X + \sqrt{1-\rho^2} W_T) - \frac{1}{2}\sigma_Y^2 T \middle| \mathbb{X}\right] \\
= \; & \frac{d\mathbb{P}}{d\mathbb{Q}} \mathbb{Q}\left[\frac{\ln\left(\frac{K-x}{y_o}\right) - \mu_Y T - \sigma_Y \rho W_T^X + \frac{1}{2}\sigma_Y^2 T}{\sigma_Y \sqrt{1-\rho^2}} \geq W_T \middle| \mathbb{X}\right] \\
= \; & \exp\left\{\frac{\mu_X}{\sigma_X} W_T^X + \frac{1}{2}\frac{\mu_X^2}{\sigma_X^2} T\right\} \left(1 - \Phi\left(\frac{\ln\left(\frac{K-x}{y_o}\right) - \mu_Y T - \sigma_Y \rho W_T^X + \frac{1}{2}\sigma_Y^2 T}{\sigma_Y \sqrt{T(1-\rho^2)}}\right)\right),
\end{aligned}$$

where *Φ* denotes the cdf of the standard normal distribution. We describe the sketch of numerical algorithm basing on a Monte Carlo approach:

1. Fix a real number $m \geq 0$ and integers $N_x > 0, N_W > 0$.
2. Draw a sample $w_1, \dots, w_{N_W}$ from the normal distribution with mean 0 and variance T.
3. For every $i = 1, \dots, N_W$ find x from the set $\{0, \frac{1K}{N_x}, \frac{2K}{N_x}, \dots, \frac{N_x K}{N_x}\}$ maximising the expression

$$\exp\left\{\frac{\mu_X}{\sigma_X}w_i + \frac{1}{2}\frac{\mu_X^2}{\sigma_X^2}T\right\}\left(1-\Phi\left(\frac{\ln\left(\frac{K-x}{y_o}\right)-\mu_Y T-\sigma_Y\rho w_i+\frac{1}{2}\sigma_Y^2 T}{\sigma_Y\sqrt{T(1-\rho^2)}}\right)\right)-mx$$

and denote it by $x^{\max}(i)$.

4. The solution is as follows: for an initial capital equal to

$$\frac{1}{N_W}\sum_{i=1}^{N_W}\exp\left\{-\frac{\mu_X}{\sigma_X}w_i - \frac{1}{2}\frac{\mu_X^2}{\sigma_X^2}T\right\}x^{\max}(i)$$

maximal expected success factor is equal to

$$\frac{1}{N_W}\sum_{i=1}^{N_W}\left(1-\Phi\left(\frac{\ln\left(\frac{K-x^{\max}(i)}{y_o}\right)-\mu_Y T-\sigma_Y\rho w_i+\frac{1}{2}\sigma_Y^2 T}{\sigma_Y\sqrt{T(1-\rho^2)}}\right)\right).$$

Different m would give a different initial capital and expected success factor.

Figure 2 shows the results of simulations for the European put option maturing at time $T = 1$ with the strike $K = 1$. The price dynamics follows 3.1 with parameters $\mu_X = \mu_Y = 0.1$, $\sigma_X = \sigma_Y = 0.3$, $Y_0 = X_0 = 1$. The diagram illustrates the dependence for different levels of ρ.

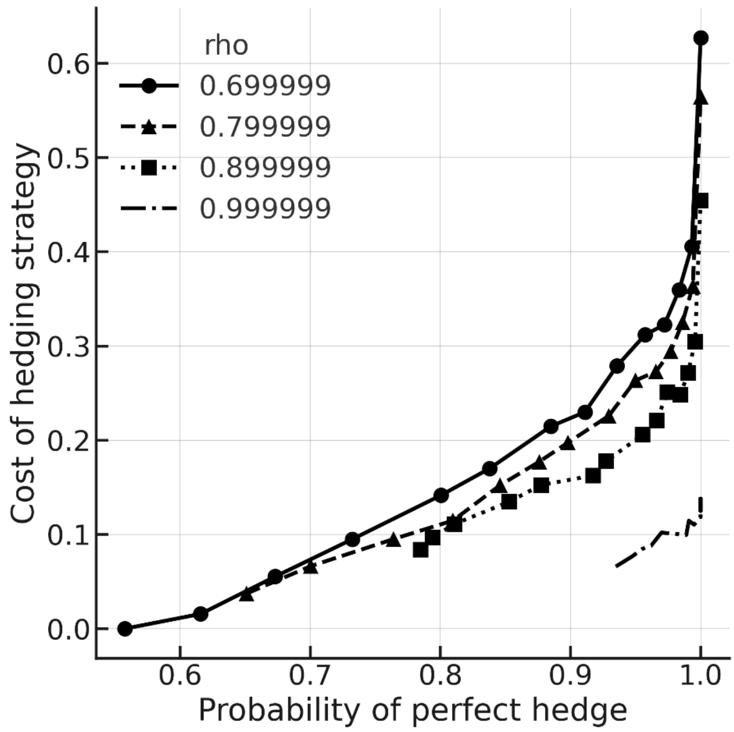

Figure 2. The results of numerical simulations for the described algorithm

Source: own elaboration.

One could verify that as expected if X almost mimics Y (i.e. ρ is almost 1), the hedging strategy with X should be very close to the hedging strategy with Y if Y was tradable. The cost of the latter is equal 0.119235 as the result of the standard Black-Scholes formula.

3.2. Applications: Solvency II

Insurance companies often neglect market risk modelling during capital modelling in the case of instruments dependent on both insurance and market risks. For instance, in the case of an equity-linked portfolio, the stochastic nature of the instrument with respect to market risks (such as stock prices) is frequently disregarded, with modelling efforts focused solely on typically insurance-related risks.

This underscores the importance of our approach, which effectively addresses this issue. Our solution provides a strategy to achieve the minimal probability of insolvency (in terms of Problem 2.1, we have $\phi_d^v := 1_{\{v \geq d\}}$ and D represents all liabilities of the insurance company at time $T = 1$). Alternatively, for a given probability (e.g., 0.995 as required by Solvency II), it determines the minimal capital required.

Unlike conventional practices that often yield non-optimal static (or nearly static) positions, our solution provides the optimal strategy, which usually requires dynamic investment in underlying assets, ensuring a more accurate and effective approach to risk management.

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References

Directive 2009/138/EC of the European Parliament and of the COUNCIL of 25 November 2009 on the taking-up and pursuit of the business of Insurance and Reinsurance (Solvency II) (recast) (*Official Journal* L 335, 17.12.2009, pp. 1-155).

Föllmer, H., & Leukert, P. (1999). Quantile Hedging. *Finance and Stochastics*, *3*(3), 251-273. https://link.springer.com/article/10.1007/s007800050062

Klusik, P., & Palmowski, Z. (2011). Quantile Hedging for Equity-Linked Contracts. *Insurance: Mathematics and Economics*, *48*(2), 280-286. https://doi.org/10.1016/j.insmatheco.2010.12.002

Klusik, P., Palmowski, Z., & Zwierz, J. (2010). Quantile Hedging for an Insider. *Probability and Mathematical Statistics*, *30*(2), 247-258. https://wuwr.pl/pms/article/view/6894/6540

Sekine, J. (2000). Quantile Hedging for Defaultable Securities in an Incomplete Market. *Surikaisekikenkyusho Kokyuroku*, (1165), 215-231. https://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/1165-13.pdf

Spivak, G., & Cvitanić, J. (1999). Maximizing the Probability of a Perfect Hedge. *Annals of Applied Probabablity*, *9*(1), 1303-1328.

Modelowanie ryzyka rynkowego w reżimie Solvency II oraz zabezpieczanie opcji bez użycia instrumentu bazowego

Streszczenie: W artykule rozwinięto narzędzia matematyczne służące do zabezpieczenia kwantylowego na rynku niezupełnym. Metody te mają dwa główne zastosowania. Po pierwsze, pozwalają obliczyć wymóg kapitałowy zgodnie z Solvency II, gdy w firmie ubezpieczeniowej występują ryzyka rynkowe i nierynkowe. Pokazujemy, jak znaleźć minimalny kapitał V_0, który zapewnia ubezpieczycielowi roczną strategię hedgingową spełniającą warunek $\mathbb{E}[\mathbf{1}_{\{V_1 \geq D\}}] = \mathbf{0.995}$, gdzie V_1 oznacza wartość aktywów przedsiębiorstwa po roku, a D jest wartością zobowiązań. Podejście uwzględnia jednocześnie ryzyka rynkowe i pozarynkowe, dzięki czemu stanowi narzędzie do wyznaczania wymogu kapitałowego SCR w ramach Solvency II. Drugim zastosowaniem jest skonstruowanie strategii zabezpieczającej dla instrumentu pochodnego bez handlu aktywem bazowym, lecz z wykorzystaniem innego aktywa, którego dynamika jest skorelowana z dynamiką aktywa bazowego (lub w inny sposób stochastycznie od niej zależna). Artykuł uogólnia wyniki uzyskane przez Klusika i Palmowskiego w 2011 roku.

Słowa kluczowe: zabezpieczenie kwantylowe, Solvency II, modelowanie kapitału, hedging opcji na aktywo niehandlowalne